Chapter 1

Temporal reachability in dynamic networks

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We construct random temporal graph models using Markov chains that preserve the random graph structure and have tunable dynamic properties. We analyze these models to determine the time it takes when starting from a random vertex to reach a large fraction of the other vertices by traversing temporal edges. The models we study are chosen for their simplicity and ability to be generalized for more complex models of threats in cybersecurity authentication systems.

1. Introduction

Dynamic network processes appear in many contexts such as spreading of infectious disease,[1] synchronization of electric power generators,[2] learning in the brain,[3] and computer communication systems.[4] The most commonly studied case is when the network itself is not changing, or only changing slowly, in time so that a static network topology is a good approximation. Given the static network the primary challenge is to discover how the dynamics, such as an epidemic outbreak, proceeds over time. Many tools and techniques have been developed to address such dynamical systems on networks.[5] A particularly important task is to relate structural properties of the network to the dynamical system progress. The opposite problem, inferring the structure of the network from a specified dynamical process, is also being addressed.[6,7]

In some situations the dynamics on the network are slower than any dy-

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namics of the network topology, or the network process itself is best viewed as a time series of edges. In this case, referred to as temporal networks, the topological changes in the network become the primary focus. Some examples of temporal networks are mobile device routing,[8] airline transportation systems,[9] and connectivity patterns in enterprise computer networks.[10] A recent survey on modeling and analysis for temporal networks can be found in the review by Holme and Saramäki.[11]

Our focus is on models for temporal networks inspired by the time series of events in centralized computer authentication systems such as those typically found in mid-size to large-size organizations.[12] Centralized computer authentication systems allow users convenient access to shared resources and applications such as printers, email, and file servers. Users request credentials with passwords, temporary pass codes, or other security methods, and then use those credentials to access networked services. The credentials are typically cached on those resources so the users can have access for some time period until the credentials are no longer valid and must be reauthorized. The credential caching is convenient for the users but it creates the risk of misuse by other users who may steal the cached credentials and use them to access parts of the network for which they are not authorized.[13,14]

The authentication system creates a stream of events of users authenticating on computers that can be represented as a temporal network. The events in which we are interested are the use of authentication credentials to connect from one computer to another. Some user authentication activities such as logins or starting a local processes only involve a single computer and a different modeling approach is required.[15] These events form time series of connections between computers and thus temporal edges in the graph of all computers in the network. The dynamical structure of connections changes rapidly on the time scale of seconds to minutes, while potential credential misuse in exploration of unauthorized parts of the network could happen on a much longer time scale of days to weeks. Thus the important dynamics are the changes in the network topology and the problem of studying the structure of authentication connections is in the category of temporal networks.

We construct and analyze two models of temporal networks with the goal of computing how the times to reach a large fraction of the vertices by traversing temporal edges changes with the structure and rate of change of the networks. Modeling the security risk of centralized computer authentication systems motivates this analysis. Although in this work we do not address fitting specific data to models, we studied the data collected in

the Los Alamos National Laboratory centralized authentication system as a basis for dynamical properties and structure for our graph modeling.[16,17] Some of the structural and dynamical properties of this data were analyzed using a graph representation.[18,19] One of the notable, and perhaps not surprising, features of the data is that the distribution of the vertex degrees is highly skewed both for the users and computers; a few users and computers have many authentication and many users and computers have few authentications. In the network of computer connections this translates to the observation that a few computers are highly connected with authentication activity and most computers are weakly connected.

A more surprising finding was that some statistical properties of the authentication graphs are stable over time even though the topological dynamics are changing rapidly. For example, the distribution of the vertex degrees is highly stable over aggregated time periods of a day even though many of the edges are changing between days. On the first 16 regular working days (Monday-Thursday are regular work days) of our example data set[16] the degree distribution is almost unchanging even though only about 60% of the edges appear on consecutive days. A similar type of dynamical network has been observed in the US air transportation route graph where the overall statistics of the network change very little even though the microdynamics are changing rapidly.[9]

To capture this type of dynamics we study and analyze two random temporal network models each based on a well-known static random graph model. The first is a temporal version of the Erdős-Rényi random graph which has been studied in the literature as "edge Markovian dynamic graphs".[20,21] The second is a generalization of that model to the Chung-Lu expected degree random graph.[22] In both cases temporal dynamics are introduced through a Markov process that preserves the random structure but changes the graph over time with an adjustable rate parameter.

The Markov condition is unlikely to be a good match for all authentication systems. The assumption ignores any burstiness that is typically seen in computer networks and that might be found on shorter timescales than the day-long time bins considered above. More detailed temporal models would need to be considered to capture those dynamics.[23] The Erdős-Rényi graph structure is clearly not a good model for the authentication data since it cannot produce graphs with skew degree distributions (the degree distribution is Poisson for large sparse graphs). However, the simplicity of the model allows us to more clearly present analysis techniques which we then apply to the temporal Chung-Lu model. The Chung-Lu model is

more appropriate for the authentication data since the degree distribution is adjustable by parameters.

For these models we address the question of how long it would take for an attacker using stolen authentication credentials to reach a large fraction of the network. We assume that when a credential is stored on a compromised computer it may be stolen and used for unauthorized access to any computer where that credential was used at some current or future time. Then from those computers other credentials can be stolen and used to reach the entire connected component of the graph containing the compromised computer. We study the impact of the structure and dynamics of the network models on the time to traverse the network. First we compute the mixing time of the induced Markov chain to show the time scales of the graph correlations in relation to the model parameters. Then we evaluate the time it would take to reach the entire network by traversing the temporal edges. Our results are asymptotic; we prove bounds for the times as the number of vertices approaches infinity.

2. Random temporal graphs

We begin with some basic definitions for temporal graphs. We consider discrete time for ease of exposition. Most of the material can easily be considered in a continuous time setting as well.

2.1. *Basic Definitions*

Let K_n denote the complete graph on n vertices.

Definition 1 (Random temporal graph)**.** A *random temporal graph* on n vertices is a probability distribution on the space $\{G_t\}_{t\geq 0}$ where $G_t \subset K_n$ for each t.

Definition 2 (Edge Markovian)**.** A random temporal graph $\{G_t\}$ on n vertices is called *edge Markovian* if there exist maps $P, Q : E(K_n) \to [0,1]$ such that for all $t \geq 0$, each edge e of G_{t+1} is determined independently with probability

$$\mathbb{P}[e \in G_{t+1}] = \begin{cases} P(e) \text{ if } e \notin G_t\,, \\ 1 - Q(e) \text{ if } e \in G_t\,. \end{cases} \tag{1}$$

Such models create a natural Markov chain, which we call the *induced Markov chain*, defined on the space of graphs on n vertices where the transi-

tion probability of moving from G_a to G_b is given by $\mathbb{P}[G_{t+1} = G_b | G_t = G_a]$. Let P be the matrix of transition probabilities between graphs on n vertices. A distribution π on all such graphs is called *stationary* if

$$\pi = \pi P.$$

Definition 3 (Mixing Time)**.** Let M be a Markov chain defined on the state space X. Let π be the stationary distribution of M. The distance $d(t)$ between the distribution of states in the chain after t steps and the stationary distribution is given by

$$d(t) = \max_x ||X_t - \pi||_{TV}\,, \tag{2}$$

where $x \in X$, and X_t denotes the distribution of states after the chain runs for t steps with an initial start of $X_0 = x$. The *mixing time* $t_{mix}(\epsilon)$ is given by

$$t_{mix}(\epsilon) = \min\{t : \ d(t) \leq \epsilon\}. \tag{3}$$

Definition 4 (Reachability)**.** The vertices $v_1, v_2, \ldots, v_k$ form a *temporal path* in $G = \{G_t\}$ if there exist times $t_1, t_2, \ldots, t_{k-1}$ such that

(1) for each $i, t_i \leq t_{i+1}$, and
(2) for each $i, (v_i, v_{i+1}) \in G_{t_i}$.

If such a path exists, we say v_k is *reachable* from v_1 within time t_{k-1}.

Note that more than one edge can be traversed at each time step so this definition allows reaching the entire connected component at time t containing a vertex visited at time t.

Given a random temporal graph G_t, a vertex u and $\delta \in [0, 1]$, what is the expected time necessary before a linear fraction δn vertices are reachable from u? To our knowledge this problem has not been studied before, and we consider it in the following sections.

There has, however, been much research into the question of "flooding times" for temporal graphs. Consider the following flow of information through a temporal graph. At time 0, one vertex is "informed." At each subsequent time, all informed vertices inform each of their current neighbors. The process stops when every vertex has been informed. The flooding time is the minimum time time necessary for every vertex to be informed. Note that the flooding time in a static graph is simply the graph diameter, while the set of vertices reachable from a vertex v in a static graph is simply the connected component containing v.

2.2. *Random Temporal Graph Models*

Most existing results are for Erdős-Rényi random temporal graphs[20,21] or, random geometric temporal graphs.[24] Erdős-Rényi random temporal graphs have appeared in the literature as "Edge-Markovian" random graphs. For instance, the flooding time is computed for a definition in which the starting graph G_0 can have any arbitrary edge set.[25] Further research considered an extension of the flooding time when vertices can only inform their neighbors in the k subsequent time steps after being themselves informed[26] and to graphs with arbitrary degree sequences.[27]

We now define a general version of this model based on the random graph structure $G(n, p)$, the Erdős-Rényi random graph on n vertices where each edge occurs independently with probability p.

Definition 5 (Erdős-Rényi random temporal graph)**.** An Erdős-Rényi random temporal graph, $G = G(n, p, \alpha)$, is an edge-Markovian random graph on n vertices defined by the parameters $p = p(n)$ and $\alpha \in [0, 1]$ such that

- $P(e) = \alpha p$,
- $Q(e) = \alpha(1 - p)$, and
- G_0 is distributed as $G(n, p)$.

As we will see, at each step in time the Erdős-Rényi temporal graph is distributed as an Erdős-Rényi random (static) graph $G(n, p)$. However the parameter α allows for the graphs G_t to be correlated in time. When $\alpha = 0$ edges are not added or removed; the graphs G_t are all equal to G_0. At the other extreme, when $\alpha = 1$, each time step is a complete resampling from $G(n, p)$.

Definition 6 (Chung-Lu random temporal graph)**.** A Chung-Lu random temporal graph $G(n, W, \alpha)$ on n vertices is defined by a distribution W on the positive reals and a function $\alpha : \mathbb{R} \to (0, 1)$. Let $W_1, W_2, \ldots, W_n$ be i.i.d. sampled from the distribution W. Let $p_{ij} = W_i W_j/(n\mathbb{E}[W])$ and $\alpha_i = \alpha(W_i)$. Then the temporal Chung-Lu model is defined as an edge-Markovian model with

- $P(v_i, v_j) = \sqrt{\alpha_i \alpha_j} p_{ij}$,
- $Q(v_i, v_j) = \sqrt{\alpha_i \alpha_j}(1 - p_{ij})$, and
- G_0 is distributed as $CL(n, \{W_i\})$, the static Chung-Lu model on n vertices with each edge (v_i, v_j) present independently with probability p_{ij}.

We assume W is bounded to ensure the p_{ij}'s are probabilities for large enough n. Our results should also apply in the more general case when asymptotically almost surely the p_{ij}'s are probabilities. However we have not worked through those details. Our definition of static Chung-Lu graphs is slightly different from that in the original paper of Chung and Lu;[22] we normalize the probabilities by $n\mathbb{E}[W]$ instead of $\sum W_i$. For large n the difference is small and this normalization is slightly more convenient in the following technical results.

The Chung-Lu model produces graphs with expected degrees given by the W_i values. The Erdős-Rényi model is a special case of the Chung-Lu model which is realized when the distribution W has all probability mass centered at one value.

3. Mixing times

The mixing time of the induced Markov chains for a random temporal graph show how the model parameters affect the time scale of correlations. After waiting for the length of the mixing time the graphs have little correlation with the starting state. The mixing is rapid, $O(\log n)$, but it can be extended to arbitrarily long times if the parameter $\alpha \to 0$.

3.1. *Erdős-Rényi random temporal graphs*

First note that each G_t is distributed as $G(n,p)$. The stationary distribution of the natural induced Markov chain is also $G(n,p)$.

Theorem 1. *Let M be the Markov chain associated with the Erdős-Rényi random temporal graph $G(n,p,\alpha)$. Suppose $p \le 1/2$. Then for every $\epsilon > 0$ there exists a C such that the mixing time is*

$$t_{mix}(\epsilon) \le \frac{\log n - \frac{1}{2}\log p + C}{-\log(1-\alpha)}. \tag{4}$$

Moreover, this is asymptotically best possible as for every ϵ, $0 < \epsilon < 1/2$ there exists a D such that

$$t_{mix}(\epsilon) \ge \frac{\log n - \frac{1}{2}\log p - D}{-\log(1-\alpha)}. \tag{5}$$

This theorem holds for $p < /1/2$, but we are mainly interested in the sparse case, i.e. when $p = c/n$ for some constant c.

We will use the following lemma.

Lemma 1. *Let X_t denote the distribution of the chain M after t steps given that X_0 is the complete graph. Then*

$$d(t) = ||X_t - \pi||_{TV}. \tag{6}$$

Moreover we have that $X_t = G(n, p_t)$ where

$$p_t = (1-\alpha)^t(1-p) + p.$$

Proof. The first statement is immediate from the definition of $d(t)$. The second statement follows from induction on t. Suppose that X_{t-1} is distributed as $G(n, p_{t-1})$. Then by definition, X_t is distributed as $G(n, q)$ where

$$\begin{aligned} q &= [1-\alpha(1-p)]\, p_{t-1} + (1-p_{t-1})\alpha p\,, \\ &= (1-\alpha)\,(p_{t-1}-p) + p\,, \\ &= (1-\alpha)\left([1-\alpha]^{t-1}\right) + p\,, \\ &= (1-\alpha)^t(1-p) + p\,. \end{aligned}$$

□

Proof of Theorem 1. To prove the upper bound set

$$t = \frac{\log n - \frac{1}{2}\log p + \omega(1)}{-\log(1-\alpha)}.$$

From Lemma 1 we have that $p_t = (1-\alpha)^t(1-p) + p$ and

$$d(t) \leq ||G(n, p_t) - G(n, p)||_{TV}.$$

Janson (in Corollary 2.12),[28] proved that if

$$n^2 \frac{(p-p_t)^2}{p} = o(1)\,,$$

holds then $||G(n, p_t) - G(n, p)||_{TV} = o(1)$. By definition,

$$p_t = \frac{\sqrt{p}}{\omega(n)}(1-p) + p\,,$$

and thus

$$n^2 \frac{(p-p_t)^2}{p} = o(1),$$

as desired. This proves the upper bound.

To prove the lower bound, let

$$t = \frac{\log n - \frac{1}{2}\log p - \omega(1)}{-\log(1-\alpha)}.$$

Let G be distributed as $G(n,p)$ and G_t as X_t, the distribution of states (graphs) after t steps when starting from the complete graph. Consider the random variables $e(G)$ and $e(G_t)$. By definition,

$$\mathbb{E}[e(G)] = \binom{n}{2}p, \tag{7}$$

while Lemma 1 implies that

$$\mathbb{E}\left[e(G_t)\right] = \binom{n}{2}\left[(1-\alpha)^t(1-p)+p\right] \geq \frac{n^2}{5}(1-\alpha)^t + \mathbb{E}\left[e(G)\right]. \tag{8}$$

Thus while $\mathbb{P}[e(G)] \leq p\binom{n}{2} = 1/2$, using the Chernoff bounds and the fact that $p \geq (1-\alpha)^t(1-p)$, we have that

$$\mathbb{P}\left[e(G_t) \leq p\binom{n}{2}\right] \leq \mathbb{P}\left[e(G_t) \leq \mathbb{E}[e(G_t)] - \frac{n}{5}(1-\alpha)^t\right], \tag{9}$$

$$\leq \exp\left(-\frac{\left(n^2(1-\alpha)^t\right)^2}{25n^2\left((1-\alpha)^t(1-p)+p\right)}\right), \tag{10}$$

$$\leq \exp\left(-\frac{n^2(1-\alpha)^{2t}}{50p}\right), \tag{11}$$

$$\leq \exp\left(-\omega(1)\right), \tag{12}$$

$$\leq o(1). \tag{13}$$

Thus $||X_t - \pi||_{TV} \geq 1/2 - o(1)$ as desired. □

3.2. *Chung-Lu random temporal graphs*

We can use similar tools to analyze mixing times in Chung-Lu random temporal graphs. To avoid confusion we point out that each choice of weights $W_1, W_2, \ldots, W_n$ defines a Markov chain on the space of n-vertex graphs. In the model $G(n, W, \alpha)$ these weights are chosen randomly from the common distribution W. Thus the model defines not only a distribution on the graphs of order n, but also a distribution on the collection of Markov chains which act on the space of n-vertex graphs. For ease of exposition, we analyze the Markov chain obtained by conditioning on the values of the W_i.

Theorem 2. *For every $0 < \epsilon < 1$ there exists a $C = C(\epsilon)$ and a $D = D(\epsilon)$ such that if $G = G(n, W, \alpha)$ is a Chung-Lu temporal graph with vertex weights $W_1, W_2, \ldots, W_n$, then the mixing time of the associated Markov chain satisfies*

$$t_{mix}(\epsilon) \leq \frac{\log n + \frac{1}{2}\log\left\langle p^{-1}\right\rangle + C}{-\log(1-m)}\ , \text{ and} \tag{14}$$

$$t_{mix}(\epsilon) \geq \frac{\log n - \frac{1}{2}\log\langle p\rangle - D}{-\log(1-\langle\alpha\rangle)}, \tag{15}$$

where

$$\left\langle p^{-1}\right\rangle = \frac{1}{\binom{n}{2}}\sum_{i<j} p_{ij}^{-1}, \tag{16}$$

$$m = \min \alpha_i, \tag{17}$$

$$\langle\alpha\rangle = \frac{\sum_{i<j}\sqrt{\alpha_i\alpha_j}}{\binom{n}{2}}, \text{ and} \tag{18}$$

$$\langle p\rangle = \frac{\sum_{i<j} p_{ij}}{\binom{n}{2}}. \tag{19}$$

Lemma 2. *Let X_t denote the distribution of the chain M after t steps given that X_0 is the complete graph. Then*

$$d(t) = ||X_t - \pi||_{TV}. \tag{20}$$

Moreover X_t, is the random graph distribution where each edge (v_i, v_j) is chosen independently with probability

$$p_{ij,t} := (1 - \sqrt{\alpha_i\alpha_j})^t\,(1 - p_{ij}) + p_{ij}.$$

Proof. The proof is analogous to the proof of Lemma 1. □

Proof of Theorem 2. Set

$$t = \frac{\log n + \frac{1}{2}\log\langle p\rangle + \omega(1)}{-\log(1-m)}.$$

We again use the work of Janson[28] which states that

$$\sum_{i,j}(p_{ij} - p_{ij,t})^2/p_{ij} = o(1)\ \Rightarrow\ ||X_t - \pi||_{TV} = o(1)\,.$$

This can be easily verified,

$$\sum_{i<j} \frac{(p_{ij} - p_{ij,t})^2}{p_{ij}} = \sum_{i<j} \frac{(1 - \sqrt{\alpha_i \alpha_j})^{2t} (1 - p_{ij})^2}{p_{ij}}, \tag{21}$$

$$\leq (1-m)^{2t} \sum_{i<j} \frac{1}{p_{ij}}, \tag{22}$$

$$= \binom{n}{2} \exp\left(2t \log(1-m) + \log\left\langle p^{-1} \right\rangle\right). \tag{23}$$

To prove the lower bound, let

$$t = \frac{\log n - \frac{1}{2}\log\langle p\rangle - \omega(1)}{\log(1 - \langle\alpha\rangle)}.$$

Let G be distributed as $G(n, p_{ij})$, the Chung-Lu graph defined by the weights $W_1, \ldots, W_n$. Let G_t be according to the law X_t: the distribution of states (graphs) after t steps when starting from the complete graph. Consider the random variables $e(G)$ and $e(G_t)$. By definition,

$$\mathbb{E}[e(G)] = \sum_{i<j} p_{ij}, \tag{24}$$

while Lemma 1 implies that

$$\mathbb{E}[e(G_t)] = \sum_{i<j} \left((1 - \sqrt{\alpha_i \alpha_j})^t (1 - p_{ij})\right) + \mathbb{E}[e(G)]. \tag{25}$$

Thus using the convexity of $f(x) = (1-x)^t$ we have that

$$\mathbb{E}[e(G_t)] - \mathbb{E}[e(G)] = \sum_{i<j} \left((1 - \sqrt{\alpha_i \alpha_j})^t (1 - p_{ij})\right), \tag{26}$$

$$\geq \binom{n}{2} \left(1 - \frac{\sum_{i<j}(1-p_{ij})\sqrt{\alpha_i\alpha_j}}{\binom{n}{2}}\right)^t, \tag{27}$$

$$\geq \binom{n}{2} (1 - \langle\alpha\rangle)^t. \tag{28}$$

Define

$$\gamma \triangleq \frac{1}{2}\binom{n}{2} (1 - \langle\alpha\rangle)^t. \tag{29}$$

Using the Chernoff large deviation inequality again we have that

$$\mathbb{P}\left[e(G) \geq \mathbb{E}[e(G)] + \gamma\right] = \mathbb{P}\left[e(G) \geq \mathbb{E}[e(G)] + \frac{1}{2}\binom{n}{2}\left(1 - \langle\alpha\rangle\right)^t\right], \quad (30)$$

$$\leq \exp\left(-\frac{\frac{1}{4}\binom{n}{2}^2\left(1 - \langle\alpha\rangle\right)^{2t}}{2\binom{n}{2}\langle p\rangle + \frac{1}{3}\binom{n}{2}\left(1 - \langle\alpha\rangle\right)^t}\right), \quad (31)$$

$$\leq \exp\left(-\frac{\binom{n}{2}\left(1 - \langle\alpha\rangle\right)^{2t}}{12\langle p\rangle}\right), \quad (32)$$

$$\leq \exp\left(-\omega(1)\right), \quad (33)$$

$$\leq o(1). \quad (34)$$

On the other hand, using the Chernoff large deviation inequality in the other direction,

$$\mathbb{P}\left[e(G_t) \geq \mathbb{E}[e(G)] + \gamma\right] = 1 - \mathbb{P}\left[e(G_t) \leq \mathbb{E}[e(G)] + \gamma\right], \quad (35)$$

$$\geq 1 - \mathbb{P}\left[e(G_t) \leq \mathbb{E}[e(G_t)] - \gamma\right], \quad (36)$$

$$\geq 1 - o(1). \quad (37)$$

Thus $||X_t - \pi||_{TV} \geq 1 - o(1)$ as desired. □

4. Reachability

Reachability measures how long it would take to traverse a large fraction of the graph starting from a given vertex. In the following we give asymptotic results for the reachability time for the Erdős-Rényi and Chung-Lu random temporal graph models. The reachability measure in Definition(4) allows traversal to the entire connected component of the graph at a given time so the analysis hinges on the sizes of the connected components in the graph. In the static versions of the models we consider there are two regimes for the connected component sizes which depend on the model parameters. In the "subcritical case" the components are all small; the reachability in the graph is controlled by connecting pathways between the components over time. In the "supercritical case" the graph has a component of size $O(n)$ and the reachability time is fast, $o(\log n)$.

4.1. *Erdős-Rényi random temporal graphs*

Theorem 3 (Subcritical case; $c < 1$)**.** *Let $p = c/n$ with $c < 1$ a constant. Let $G = TG(n, p, \alpha)$ be an Erdős-Rényi random temporal graph and v, a vertex of G. Then there exists a constant ρ depending only on c and α such*

that for any $0 < \epsilon < 1$*, asymptotically almost surely the time it takes to reach* $(1-\epsilon)n$ *vertices is* $\rho \log n + o(\log n)$.

Theorem 4 (Supercritical case; $c > 1$)**.** *Let* $p = c/n$ *with* $c > 1$*. Then for any* $0 < \epsilon < 1$*, asymptotically almost surely the time it takes to reach* $(1-\epsilon)n$ *vertices is* $o(\log n)$.

An upper bound on the order of magnitude in Theorem 4 has already appeared in the literature.[25] We supply an independent proof and determine the exact value of ρ.

Define the random variables $X_t, t \geq 0$ such that X_t is the number of vertices reachable from v within time t. We remark that for all $t, X_t \geq X_{t-1}$ and that X_0 is the size of the component of G_0 containing v. To bound the X_t, we first consider a truncated version of X_t that will be useful for technical reasons in the subsequent analysis. Fix a constant $\epsilon_0 > 0$ and define

$$\hat{X}_t \triangleq \min\{\epsilon_0 n, X_t\}. \tag{38}$$

To analyze the growth of $\hat{X}_t$, we construct a growth process $\mathcal{B} = \mathcal{B}(N, p, \alpha)$ described as follows. The process starts with one individual and the number of individuals at time t of this process is given by Z_t. Suppose v_i is one of the Z_t individuals at time t. At each time step t, every such individual v_i gives birth to $Y_{t;i}$ children where the $Y_{t;i}$ are i.i.d. from $Bin(N, \alpha p)$. Then each of the new progeny have further progeny $Z_{t;i;j}$ sampled i.i.d. from $C = CG(N, p)$ - the distribution of component sizes in $G(N, p)$. Formally for each of the $Y_{t;i}$ progeny of v_i sample $Z_{t;i;j}$ i.i.d. from $C = CG(N, p)$ where $j = 1, 2, \ldots Y_{t;i}$. Each such sampling is considered independently over all realizations of $G(N, p)$. The process starts with $Z_0 = 1$. Note that as individuals do not die we have

$$Z_t = \sum_{i=1}^{Z_{t-1}} \left(1 + \sum_{j=1}^{Y_{t;i}} (Z_{t;i;j}) \right). \tag{39}$$

The growth process $\mathcal{B}(N, p, \alpha)$ described above is designed to mimic the growth process associated with the reachability set as stated in the following lemma.

Lemma 3. *Fix* $\epsilon_0 > 0$*. Let* $\hat{X}_t$ *be the truncated process defined in (38). Let* $Z_t(n)$ *be the number population size at level* t *in* $\mathcal{B}(n, p, \alpha)$ *and let* $Z_t(n(1-\epsilon_0))$ *be the same quantity in* $\mathcal{B}(n(1-\epsilon_0), p, \alpha)$*. Define* $\hat{Z}_t(n(1-$

$\epsilon_0)) = \min\{\epsilon_0 n, Z_t(n(1-\epsilon_0))\}$. *Then* $\hat{X}_t$ *is stochastically bounded below by* $\hat{Z}_t(n(1-\epsilon_0))$ *and stochastically bounded above by* $Z_t(n)$.

We expect the process $\mathcal{B}$ to grow roughly at the rate of $(1+\eta)$ for some positive η at each time step. We also expect that for large t, Z_t should be close to $(1+\eta)^t$. However this does not hold for small t. Define the random variable

$$W = \sum_{j=1}^{Y_{t;i}} Z_{t;i;j}, \tag{40}$$

and let

$$\eta \triangleq \mathbb{E}[W] = \alpha \frac{np}{1-np}, \tag{41}$$

be its mean. Note that Equation (39) describes the evolution of the number of individuals at depth t in a Galton-Watson branching process $\mathcal{T}$ with offspring distribution described by the random variable $1+W$. With this equivalence, we can now use the well-studied theory of Galton-Watson branching process to analyze the number of individuals Z_t in the growth process $\mathcal{B}$. We first state the following results from branching process theory.

Lemma 4. *Let* $m \triangleq 1 + \mathbb{E}[W] = 1 + \eta$ *be the mean number of offsprings in the branching process* $\mathcal{T}$. *Let* Y_N *denote the number of vertices at depth* N *in* $\mathcal{T}$. *Then there exists a random variable* $\mathbf{Y}$ *such that*

$$\lim_{N\to\infty} m^{-N} Y_N = \mathbf{Y} \text{ with probability 1}. \tag{42}$$

Additionally we have

$$\mathbb{P}\left[Y_N = 0\right] = p_{ext} + o_N(1), \tag{43}$$

where p_{ext} *is the extinction probability of the branching process* $\mathcal{T}$, *and*

$$\mathbb{P}\left[0 < Y_N < y^N\right] \leq \mathbb{P}\left[0 < \mathbf{Y} < (y/m)^N\right](1 + o_N(1)), \tag{44}$$

where $o_N(1)$ *is a term that converges to* 0 *as* $N \to \infty$.

Proof. The statement in (42) is a well-known result in branching process theory.[29] Both Equations (43) and (44) can be proved directly by using $K_N = y^N$ (using the notation in the reference) in Corollary 5 in Fleischmann and Wachtel[30] and observing that $\mathbb{E}[(1+W)\log(1+W)] < \infty$. □

Now we can define the quantity ρ in the statement of Theorem 3 as

$$\rho = \frac{1}{\log(1+\eta)}\,. \tag{45}$$

In view of Lemma 3, we first prove the statements in Theorem 3 for the growth process $\mathcal{B}$ before proceeding to prove the theorem itself.

Lemma 5. *Let $0 < \bar{\epsilon} < \epsilon_0$. Let $T = \inf\{t : \mathcal{B}_t(n,p) \geq \bar{\epsilon}n\}$. Then for any $0 < \gamma < \rho$, we have asymptotically almost surely*

$$(\rho-\gamma)\log n \leq T \leq (\rho+\gamma)\log n. \tag{46}$$

Proof. From the previously stated equivalence between the growth process $\mathcal{B}$ and the branching process $\mathcal{T}$, we have $Z_t = Y_t$. Note that $\mathbb{E}[Z_t] = (1+\eta)^t$. Using the Markov inequality, we have

$$\mathbb{P}[Z_{(\rho-\gamma)\log n} > \bar{\epsilon}n] \leq \frac{(1+\eta)^{(\rho-\gamma)\log n}}{\bar{\epsilon}n} = \frac{(1+\eta)^{-\gamma\log n}}{\bar{\epsilon}} = o(1).$$

This shows that asymptotically almost surely $T \geq (\rho-\gamma)\log n$.

To obtain a bound from the other direction,

$$\begin{aligned}
\mathbb{P}\left[Z_{(\rho+\gamma)\log n} < \bar{\epsilon}n\right] &= \mathbb{P}\left[Y_{(\rho+\gamma)\log n} = 0\right] + \mathbb{P}\left[0 < Y_{(\rho+\gamma)\log n} < \bar{\epsilon}n\right]\,, \\
&\leq p_{ext} + \mathbb{P}\left[\mathbf{Y} \in \left(0, \frac{\bar{\epsilon}n}{(1+\eta)^{(\rho+\gamma)\log n}}\right)\right](1+o(1))\,, \\
&= \mathbb{P}\left[\mathbf{Y} \in \left(0, \bar{\epsilon}(1+\eta)^{-\gamma\log n}\right)\right](1+o(1))\,, \qquad (47) \\
&= o(1)\,.
\end{aligned}$$

□

It remains to prove Lemma 3.

Proof of Lemma 3. To analyze the growth of X_t observe that when moving from time t to $t+1$ the deletion of edges present in time t does not have any effect on the reachable vertices. The only part that increases the size of the reachable set is the addition of previously absent edges, which happens with probability αp independently. The newly reachable vertices are those vertices which are now adjacent to at least one of the previous X_t vertices and the components of these newly adjacent vertices in G_{t+1}. The number of newly adjacent vertices is distributed as $\mathcal{B}(n-X_t, \alpha p)$ which is stochastically dominated by $\mathcal{B}(n, \alpha p)$. Further, the size of the components of the new neighbors are distributed as the component size distribution in $G(n-X_t, p)$ and the total number of vertices in these components is bounded above using the union bound by the sum of the component sizes.

This proves that X_t is stochastically bounded above by the population at time t in $\mathcal{B}(n,p)$. Since we have $\hat{X}_t \leq X_t$, the stochastic dominance relation also holds for $\hat{X}_t$.

To show the lower bound, we observe that since $\hat{X}_t \leq \epsilon_0 n$ by definition the component sizes of the newly adjacent vertices is stochastically dominates the component sizes in $G(n(1-\epsilon_0),p)$. However, there might be intersection between the newly added components which makes it unclear whether $\hat{X}_t$ stochastically dominates $\hat{\mathcal{B}}$. To get past this issue, we reveal each of the newly added components sequentially. Let $C_1,\ldots,C_{\hat{X}_t}$ be the newly added components at time t for $\hat{X}_t$. Let $D_1,\ldots,D_{Z_t}$ be the newly added number of children at time t in $\hat{\mathcal{B}}(n(1-\epsilon_0),t)$. We can assume as an induction hypothesis that $\hat{X}_t$ stochastically dominates $\hat{\mathcal{B}}(n(1-\epsilon_0),t)$. We need to prove that $|\bigcup_{i=1}^{\hat{X}_t} C_i|$ stochastically dominates $|\bigcup_{i=1}^{\hat{\mathcal{B}}(n(1-\epsilon_0),t)} D_i|$. We can write

$$|\bigcup_{i=1}^{\hat{X}_t} C_i| = \sum_{i=1}^{\hat{X}_t} |C_i \setminus (\bigcup_{j=1}^{i-1} C_j)|.$$

Note that the quantity $|C_i \setminus (\bigcup_{j=1}^{i-1} C_j)|$ is distributed as component sizes in $G(n-\hat{X}_t,p)$ which stochastically dominates the component sizes in $\mathcal{B}(n(1-\epsilon_0),p)$, which is in turn identical to $|D_i|$. The proof then follows by induction. □

We next state another result which says that its takes a small amount of additional time to reach (almost) all vertices once we have reached a linear fraction of them.

Lemma 6. *Let $0<\epsilon,\bar{\epsilon}<1$. Suppose that $X_{t_0}\geq \bar{\epsilon}n$. Let $T_{\bar{\epsilon},1-\epsilon}$ be the additional time it takes to reach at least $(1-\epsilon)n$ vertices. Then asymptotically almost surely we have $T_{\bar{\epsilon},1-\epsilon}=o(\log n)$.*

Proof. Let R_t be the number of unreached vertices at time t_0+t, i.e. , $R_t = n - X_{t_0+t}$. Let v be any vertex among the R_t unreached vertices. Then the probability that v will be reached at time t_0+t+1 is at least $1-(1-c\alpha/n)^{\bar{\epsilon}n} \geq 1-0.5e^{-c\alpha\bar{\epsilon}} \triangleq \bar{\delta} > 0$. Hence the number of unreached vertices at time t_0+t+1 is stochastically dominated by $Bin(R_t, 1-\bar{\delta})$. This allows us to recursively define an overestimating process for the number of unreached vertices at any time t_0+t. Let $\hat{R}_0 = (1-\bar{\epsilon})n$, and let $\hat{R}_{t+1} = Bin(\hat{R}_t,\bar{\delta})$. From the previous discussion, for any $t\geq 0$ by induction $\hat{R}_t$ stochastically dominates R_t. Also by induction, $\hat{R}_t$ is distributed as $Bin(1-\bar{\epsilon}n,(1-\bar{\delta})^t)$.

Let $f(n) \to \infty$ be such that $f(n) = o(\log n)$. By Markov inequality

$$\mathbb{P}\left[]R_{f(n)} > \epsilon n\right] \leq \mathbb{P}\left[\hat{R}_{f(n)} > \epsilon n\right] \leq \frac{(1-\bar{\epsilon})}{\epsilon}(1-\bar{\delta})^{f(n)} = o(1).$$

This shows that $T_{\bar{\epsilon},1-\epsilon} = o(\log n)$. □

We now have all the results required to complete the proof of Theorem 3.

Proof of Theorem 3. Fix $\epsilon_0 > 0$ and let $0 < \bar{\epsilon} = \epsilon_0/2$. Let $T_{\epsilon_0/2} = \inf\{t : \mathcal{B}_t(n,p) \geq \bar{\epsilon}n\}$. Let $\gamma > 0$ be arbitrary. From Lemma 3 we have that X_t is stochastically bounded above by $Z_t(n)$. Combined with Lemma 5 we conclude that asymptotically almost surely $T_{\epsilon_0/2} \geq (\rho - \gamma)\log n$.

Let $\eta(\epsilon_0) \leq \eta$ be the quantity corresponding to η in $G(n(1-\epsilon_0), \alpha p)$. Note that $\lim_{\epsilon_0 \to 0} \eta(\epsilon_0) = \eta$. Let

$$\rho \leq \rho(\epsilon_0) \triangleq \frac{1}{\log(1+\eta(\epsilon_0))}. \tag{48}$$

Again from Lemma 3 we have that X_t is stochastically bounded below by $Z_t(n(1-\epsilon_0))$. Combining with Lemma 5 we conclude that asymptotically almost surely $T_{\epsilon_0/2} \leq (\rho(\epsilon_0) + \gamma)\log n$. Restating, asymptotically almost surely, T satisfies

$$(\rho - \gamma)\log n \leq T_{\epsilon_0/2} \leq (\rho(\epsilon_0) + \gamma)\log n. \tag{49}$$

Let $T_{1-\epsilon}$ be the time it takes to reach at least $(1-\epsilon)n$ vertices. Then $T_{1-\epsilon} = T_{\epsilon_0/2} + T_{\epsilon_0/2,1-\epsilon}$. Then combining (49) and Lemma 6 we have that

$$(\rho - \gamma)\log n + o(\log n) \leq T_{1-\epsilon} \leq (\rho(\epsilon_0) + \gamma)\log n + o(\log n). \tag{50}$$

Notice that (50) holds for all $\gamma > 0$ and all $\epsilon_0 > 0$. So letting both γ and ϵ_0 tend to zero and using the fact that $\lim_{\epsilon_0 \to 0} \rho(\epsilon_0) = \rho$ we have that $T_{1-\epsilon} = \rho \log n + o(\log n)$. □

We devote the rest of this section to the proof of Theorem 4 in the supercritical case. First we state a well-known fact regarding the giant component in Erdős-Rényi random graphs.

Lemma 7. *Let $G(n,p)$ be an Erdős-Rényi random graph with $p = c/n$ and $c > 1$. Then there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ depending on c such that, there is a connected component of size at least $\delta_1 n$ in G with probability at least $1 - e^{-\delta_2 n}$.*

Proof of Theorem 4. Let $f(n)$ be such that $f(n) \to \infty$ and $f(n) = o(\log n)$. The probability that $G_0, \ldots, G_{f(n)}$ all have a connected component of size at least $\delta_1 n$ is greater than $1 - f(n)e^{-\delta_2 n} = 1 - o(1)$. For $0 \leq t \leq f(n)$, the probability that in G_t the vertex v is not connected to the giant component is less than $(1 - \alpha c/n)^{\delta_1 n} \leq e^{-\alpha c \delta_1}$. Thus, the probability that the size of the reachable set at time $t = f(n)$ is smaller than $\delta_2 n$ is bounded above by the probability that the vertex v does not belong to the giant component in any of $G_1, \ldots G_{f(n)}$, which in turn is bounded above by $e^{-c\alpha\delta_1 f(n)} = o(1)$.

Additionally, by Lemma 6, the time it takes to reach $1 - \bar{\epsilon}n$ vertices starting from $\delta_1 n$ vertices is $o(\log n)$. Together this completes the proof. □

4.2. *Chung-Lu random temporal graphs*

Theorem 5 (Subcritical case)**.** *Let $G = G(n, W, \alpha)$ be a Chung-Lu random temporal graph with $\mathbb{E}[W^2] < \mathbb{E}[W]$. Let v be a vertex of G. Then there exists a constant ρ such that for each $\epsilon > 0$, asymptotically almost surely*

$$T_{1-\epsilon} \leq \rho \log n + o(\log n),$$

where $T_{1-\epsilon}$ is the time required to reach $(1-\epsilon)n$ vertices from v and

$$\rho = \left(1 + \frac{\mathbb{E}\left[\alpha(W)W^2\right]}{\mathbb{E}[W]} + \frac{\mathbb{E}\left[\sqrt{\alpha(W)}W^2\right]^2}{\mathbb{E}[W]^2(1 - \mathbb{E}[W^2]/\mathbb{E}[W])}\right)^{-1}.$$

We conjecture that, just as in the Erdős-Rényi case, the value of ρ is exact. However we do not prove the lower bound here as the technical details are more complex.

Theorem 6 (Supercritical case)**.** *Let $G = G(n, W, \alpha)$ be a Chung-Lu random temporal graph with $\mathbb{E}[W^2] > \mathbb{E}[W]$. Suppose that the support of W and $\alpha(W)$ do not contain 0. Let v be a vertex of G. Then asymptotically almost surely*

$$T_{1-\epsilon} = o(\log n),$$

where $T_{1-\epsilon}$ is the time required to reach $(1-\epsilon)n$ vertices from v.

The proofs of the two theorems follow the same general outlines of the proofs in the Erdős-Rényi case. Many of the steps are even exactly the same. Thus we often refer to the Erdős-Rényi case and leave out those details which are same. Instead we devote our attention to outlining and explaining the differences in the two sets of proofs.

It is well known that when $\mathbb{E}[W^2] < \mathbb{E}[W]$, asymptotically almost surely, the Chung-Lu random graph $CL(n, W)$ has no giant component while when $\mathbb{E}[W^2] > \mathbb{E}[W]$ there is a unique component of size linear in n.[22] When $\mathbb{E}[W^2] < \mathbb{E}[W]$ the graph is called subcritical. Janson and Riordan have shown[31] that in this regime the average component has size is

$$\mathbb{E}[|C(v)|] = 1 + \frac{\mathbb{E}[W]}{1 - \mathbb{E}\left[W^2\right]/\mathbb{E}[W]}\,, \tag{51}$$

where $C(v)$ denotes the component containing the vertex v. The above result is obtained by averaging over the vertices uniformly. We will need a related result in which the initial vertex is chosen with probability proportional to its weight.

We now proceed in a similar manner as in the proof of Theorem 3. Define the growth process $\mathcal{B} = \mathcal{B}(n, W, \alpha)$ as follows. The process starts with one individual; the number of individuals at time t in of the process is given by Z_t. The individuals in the process are distinguished by their types: v_i has type W_i. Each time step t consists of two parts, denoted (a) and (b).

In part (a) of time step t, every individual v_i in the process gives birth to $Y_{t;i} = \sum_{j=1}^{n} X_{t;i,j}$ children where the $X_{t;i,j}$ are Bernoulli random variables chosen independently with probability $\sqrt{\alpha_i \alpha_j} p_{ij}$. Let μ_W be the probability measure of the distribution W. Then we can write $\int d\mu_W(x) = 1$. Note that the collection of children at time t have types which are distributed according to law $\alpha\mathcal{W}$ defined by the probability measure $x\alpha(x)d\mu_W(x)/\mathbb{E}[\alpha W]$ since a child of type W_i is chosen with probability proportional to $W_i\alpha(W_i)$.

In part (b) of a time step each of the $Y_{t;i}$ children at time t have further progeny. We define, for each type W_j, $C = CG(n, W; W_j)$, the distribution of the random variable $|C(v)| - 1$ defined over all realizations of the random graph $CL(n, W)$ in which v is given the weight W_j.

Now in part (b) of the process a child v_j of type W_j has $Z_{t;i;j}$ further progeny sampled independently from $C = CG(n, W; W_j)$. Note that types of these $Z_{t;i;j}$ further progeny are selected randomly with probabilities proportional to value of their type W_k. In other words, the types of the $Z_{t;i;j}$ individuals are distributed according to the law $\mathcal{W}$ defined by the probability measure $xd\mu_W(x)/\mathbb{E}[W]$. Importantly, as v_j was born in part (a), its type is distributed according to the law $\alpha\mathcal{W}$ defined by the probability measure $x\sqrt{\alpha(x)}d\mu_W(x)/\mathbb{E}[\alpha W]$. This is because by the definition of part (a), the type W_j is selected with probability proportional to $\sqrt{\alpha(W_j)}W_j$.

As before we can inductively define the Z_t,

$$Z_t = \sum_{i=1}^{Z_{t-1}} \left(1 + \sum_{j=1}^{Y_{t;i}} Z_{t;i;j} \right). \tag{52}$$

In the sum above we assume, for ease of exposition, that the indices correspond to the types of the respective individuals in the process. We have described the number of Z_t of individuals at each step in the process. Let the distribution of their types at time t be $\mathcal{D}_t$.

Lemma 8. *Let v be a vertex chosen at uniform in the random temporal graph $G = G(n, W, \alpha)$. For $t \geq 0$, define X_t to be the number of vertices reachable from v within time t. Let Z_t be the population size of the process $\mathcal{B}(n, W, \alpha)$. Then X_t is stochastically bounded from above by Z_t.*

The proof is essentially the same as the first part of the proof of Lemma 3 and is left as an exercise for the interested reader.

The proof of Theorem 5 is similar to the proof of the upper bound for Theorem 3. In particular, the process $\mathcal{B}$ in transformed into a Galton-Watson branching process. Equation (52) describes the evolution of the number of individuals in a multi-type branching process at depth t with offspring distributed as $\mathcal{D}_t$ and the number of offspring described by the random variable $Q_t + 1$ where

$$Q_t = \sum_{j=1}^{Y_{t;i}} Z_{t;i;j} \,. \tag{53}$$

Let $\eta(t)$ be the expected value of Q_t. A technical result[32] states that if m_t is the number of individuals in such a multi-type branching process at time t then $m_t / \prod_t (1 + \eta(t))$ will converge to a random variable U which is zero with probability zero. This is because the branching cannot die out.

To determine the rate of growth of m_t, it is convenient to split it into two parts.

Lemma 9. *Let A_t and B_t be the expected number of individuals whose types are distributed according to the laws $\alpha\mathcal{W}$ and $\mathcal{W}$, respectively. Then*

$$m_t - 1 = A_t + B_t \,. \tag{54}$$

Furthermore, there exist constants η and λ such that

$$A_t = (1 + \eta)^t \,, \text{ and} \tag{55}$$

$$B_t = \lambda A_t \,. \tag{56}$$

Lemma 10. *Let $0 < \bar{\epsilon}, \epsilon < 1$. Suppose that $X_{t_0} \geq \bar{\epsilon} n$. Let $T_{\bar{\epsilon},1-\epsilon}$ be the additional time it takes to reach at least $(1-\epsilon)n$ vertices. Then asymptotically almost surely we have $T_{\bar{\epsilon},1-\epsilon} = o(\log n)$.*

Proof of Theorem 5. By Lemma 8 we know that the graph reachability process X_t is stochastically dominated by Z_t. We also have that Z_t in expectation grows exponentially.

We can use a large deviation inequality as in Inequality (47) to show that the time for the process $\mathcal{B}$ to reach at least ϵn vertices is asymptotically almost surely no more than $\rho + \gamma$ where

$$\rho = \frac{1}{1+\eta}, \tag{57}$$

and $0 < \gamma < \rho$.

As γ can be chosen arbitrarily small and as the number of discovered individuals in $\mathcal{B}$ at time t stochastically dominates X_t we have in the limit that asymptotically almost surely $T_\epsilon \leq \rho \log n$.

Finally Lemma 10 states that once $\bar{\epsilon} n$ vertices are reachable from v, the set of reachable vertices starts to grow very quickly. At this point only $o(\log n)$ more time steps are necessary for all $(1-\epsilon)n$ vertices to be reachable from v. □

Proof of Lemma 10. Let R be the set of vertices reachable from v in time t_0 so $|R| \geq \bar{\epsilon} n$. Let U be the rest of the vertices. Since $\alpha \geq 0$ and W is defined on the positive reals, there exist m_W, m_α, and δ such that asymptotically almost surely the set $\bar{U} = \{v_i \in U : W_i \geq m_W \text{ and } \alpha_i \geq m_\alpha\}$ satisfies $|U \backslash \bar{U}| \leq \epsilon n/2$. Indeed we can choose m_α small enough that there exist a δ such that the set $\bar{R} = \{v_i \in R : \alpha_i \geq m_\alpha\}$ satisfies

$$\sum_{v_i \in \bar{R}} W_i \geq \delta n \,. \tag{58}$$

Let $\bar{U}_t$ be the number of unreached vertices of $\bar{U}$ at time $t_0 + t$. Let v_i be any vertex in U_t. Then the probability that v_i will be reached at time

$t_0 + t + 1$ is at least

$$1 - \prod_{v_j \in \bar{R}} (1 - \sqrt{\alpha_i \alpha_j} p_{ij}) \geq 1 - \exp\left(\sum_{v_j \in \bar{R}} -\sqrt{\alpha_i \alpha_j} p_{ij} \right), \tag{59}$$

$$\geq 1 - \exp\left(-\frac{\sqrt{\alpha_i} W_i}{n\mathbb{E}[W]} \sum_{v_j \in \bar{R}} \sqrt{\alpha_j} W_j \right), \tag{60}$$

$$\geq 1 - \exp\left(-\frac{\sqrt{m_\alpha} m_W}{n\mathbb{E}[W]} \sqrt{m_\alpha} \delta n \right). \tag{61}$$

Letting $\bar{\delta} \triangleq 1 - \exp\left(-m_\alpha m_W \delta / \mathbb{E}[W]\right)$ we have that $\bar{U}_t$ is stochastically dominated by $\hat{U}_t$ distributed as $Bin(n(1-\epsilon/2), (1-\delta)^t)$. Let $f(n) \to \infty$ with $f(n) = o(\log n)$. By the Markov inequality,

$$\begin{aligned} \mathbb{P}[U_{f(n)} > \epsilon n] &\leq \mathbb{P}[\bar{U}_{f(n)} > \epsilon n/2], \\ &\leq \mathbb{P}[\hat{U}_{f(n)} > \epsilon n/2], \\ &\leq \frac{(1-\bar{\epsilon})}{\epsilon/2} (1-\bar{\delta})^{f(n)} = o(1). \end{aligned}$$

This shows that $T_{\bar{\epsilon}, 1-\epsilon} = o(\log n)$. □

Lemma 11. *Let v be an individual in the process $\mathcal{B}(n, W, \alpha)$ discovered (born) at time t with weight (i.e. type) distributed according to the probability measure ν. Let $Y_{\tau;v}$ denote the number of children of v in the first part of time step τ for $\tau > t$. Then*

$$\mathbb{E}[Y_{t+1;v}] = \frac{\int \sqrt{\alpha(x)\alpha(y)} xy}{\mathbb{E}[W]} d\mu_W(x) d\nu(y). \tag{62}$$

If u was discovered in part (a) of time step t and is distributed according to ν_1, then in part (b) of that time step u has $Z_{t,u}$ further progeny with

$$\mathbb{E}[Z_{t,v}] = \frac{\int x \, d\nu_1(x)}{1 - \mathbb{E}[W^2]/\mathbb{E}[W]}, \tag{63}$$

many further progeny.

As the new progeny in part (b) of the process correspond to vertices in a sampled component, the second part of the lemma is thus equivalent to claiming that the sampled component containing v has size in expectation

$$1 + \frac{\int x \, d\nu(x)}{1 - \mathbb{E}[W^2]/\mathbb{E}[W]}.$$

Proof. The first part of the lemma is follows from the fact that a new neighbor v_j in part (a) of the process are discovered with probability

$$\sqrt{\alpha_i \alpha_j} W_i W_j / n\mathbb{E}[W]\,,$$

when v has type W_i.

The second part of the lemma follows from results on the average component sizes in inhomogeneous random graphs[31]. □

Proof of Lemma 9. From Lemma 11 there exist constants γ_1, γ_2, and γ_3 such that

$$A_t = (\gamma_1 + 1)A_{t-1} + \gamma_2 B_{t-1}, \tag{64}$$

$$B_t = \gamma_3(A_t - A_{t-1}) + B_{t-1}. \tag{65}$$

Using the fact that $B_1 = \gamma_3 A_1$, and solving we have that $\eta = \gamma_1 + \gamma_2\gamma_3$. □

The values of the constants γ_1, γ_2, and γ_3 can be directly computed using Lemma 11. The are as follows:

$$\gamma_1 = \int \frac{xy\sqrt{\alpha(x)\alpha(y)}}{\mathbb{E}[W]} \frac{x\sqrt{\alpha(x)}d\mu_W(x)d\mu_W(y)}{\mathbb{E}[\sqrt{\alpha(W)}W]} = \frac{\mathbb{E}[\alpha(W)W^2]}{\mathbb{E}[W]}, \tag{66}$$

$$\gamma_2 = \int \frac{xy\sqrt{\alpha(x)\alpha(y)}}{\mathbb{E}[W]} \frac{xd\mu_W(x)d\mu_W(y)}{\mathbb{E}[W]} = \frac{\mathbb{E}[\sqrt{\alpha(W)}W^2]\mathbb{E}[\sqrt{\alpha(W)}W]}{\mathbb{E}[W]^2}, \tag{67}$$

$$\gamma_3 = \frac{\int x\sqrt{\alpha(x)}xd\mu_W(x)/\mathbb{E}[\sqrt{\alpha(W)}W]}{1 - \mathbb{E}[W^2]/\mathbb{E}[W]} = \frac{\mathbb{E}[\sqrt{\alpha(W)}W^2]}{\mathbb{E}[\sqrt{\alpha(W)}W]\,(1 - \mathbb{E}[W^2]/\mathbb{E}[W])}\,. \tag{68}$$

This concludes the proof of Theorem 5.

To prove Theorem 6 we will use the following well-known result.

Lemma 12. *Let $G = CL(n, W)$ be a Chung-Lu random graph with $\mathbb{E}[W^2] \geq \mathbb{E}[W]$. Then there exist constants $\delta_1 > 0$ and $\delta_2 > 0$, depending only on W, such that there is a connected component in G with vertex weights that sum to at least $\delta_1 n$ with probability at least $1 - \exp(-\delta_2 n)$.*

Proof of Theorem 6. As the supports of W and $\alpha(W)$ do not contain 0, there exist a nonzero m_W and m_α such that each vertex in the graph has weight at least m_W and each $\alpha_i \geq m_\alpha$. By Lemma 12, there exist nonzero δ_1, δ_2 such that with probability at least $1 - \exp(-\delta_2 n)$, there is a connected component, C_1, in G_t such that $\sum_{v_i \in C_1} W_i \geq \delta_1 n$. Let v be a vertex. Then the conditioning on the existence of a giant component in

G_t, the probability v is not connected to the giant component in step t is at most $(1 - m_\alpha m_W \delta_2)/\mathbb{E}[W])$ which is a constant independent of n.

The rest of the proof is the same as in Theorem 4. □

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